Kappa Distributions in the Language of Superstatistics

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Comisión Chilena de Energía Nuclear
Ministerio de Energía

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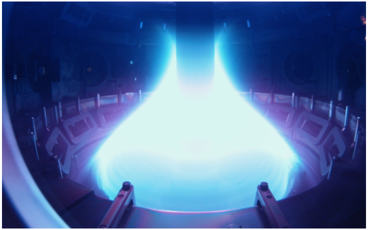
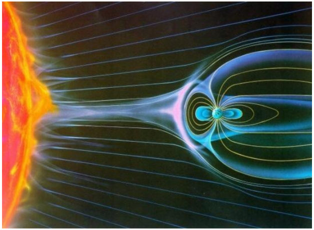
Outline of the talk

- Non-Maxwellian distributions
- Kappa distributions in connection with Tsallis statistics
- Inconsistencies in the use of Tsallis entropy
- An alternative: Superstatistics
- Obtaining kappa distributions from superstatistics
- Departure from equilibrium in kappa-distributed plasmas

How to describe non-equilibrium systems?

Laboratory and space plasmas **do not always follow** the predictions from equilibrium thermodynamics

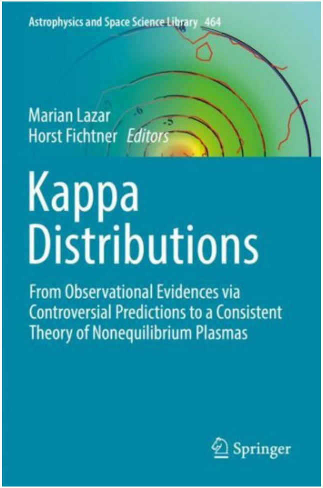
 

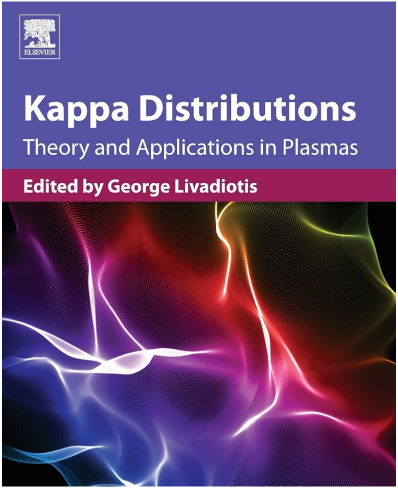
Example:
Electron (also ions?) velocities cannot be described by the Maxwellian distribution,

$$P(\boldsymbol{v}|m, T) = \left(\sqrt{\frac{m}{2\pi k_B T}}\right)^3 \exp\left(-\frac{m\boldsymbol{v}^2}{2k_B T}\right)$$

as one would expect for equilibrium systems.

Kappa distributions





The kappa distribution of velocities

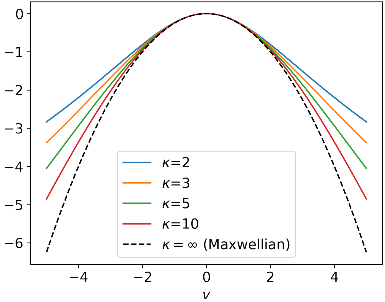
The kappa distribution for the velocity v of a particle in a plasma* can be written as

$$P(v|\kappa, v_{\text{th}}) = \frac{1}{\eta_\kappa(v_{\text{th}})}\left[1 + \frac{1}{\kappa - \frac{3}{2}}\frac{v^2}{v_{\text{th}}^2}\right]^{-(\kappa+1)}$$

Here, κ is the *spectral index* and v_{th} is the so-called *thermal velocity*, such that

$$\frac{m v_{\text{th}}^2}{2} = k_B T \tag{1}$$

provides a definition of temperature T.



The limit $\kappa \to \infty$ of the kappa distribution is the Maxwellian distribution in equilibrium,

$$P(v|T) = \left(\sqrt{\frac{m}{2\pi k_B T}}\right)^3 \exp\left(-\frac{m v^2}{2 k_B T}\right). \tag{2}$$

*G. Livadiotis and D. J. McComas. Astrophys. J. **741**, 88 (2011).

The kappa distribution as a q-Maxwellian distribution

By defining the q-exponential function

$$\exp_q(x) := \Big[1 + (1-q)x\Big]_+^{\frac{1}{1-q}}$$

such that

$$\lim_{q \to 1} \exp_q(x) = \exp(x),$$

we can conveniently write the kappa distribution as a q-Maxwellian,

$$P(\boldsymbol{v}|\kappa, v_{\text{th}}) = \frac{1}{Z_q(T_0)} \exp_q\left(-\frac{m\boldsymbol{v}^2}{2k_B T_0}\right), \tag{3}$$

provided that we set $q = 1 + \frac{1}{\kappa+1}$ and $k_B T_0 = \left(\frac{\kappa - \frac{3}{2}}{\kappa+1}\right) k_B T.$

In the limit $\kappa \to \infty$ we see that $q \to 1$ and $T_0 \to T$ as expected.

Is the kappa distribution a maximum entropy distribution?

The maximum of the non-extensive (Tsallis) entropy*

$$\mathcal{S}_q[p] := \frac{1}{q-1}\left(1 - \int dv\, p(v)^q\right)$$

subject to the constraints on the *escort expectation*

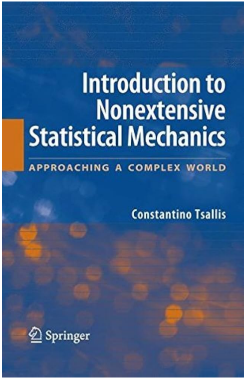
$$\frac{\int dv\, p(v)^q \left(\frac{mv^2}{2}\right)}{\int dv\, p(v)^q} = \bar{k} \qquad (4)$$

and normalization,

$$\int dv\, p(v) = 1, \qquad (5)$$

is precisely the kappa distribution (q-Maxwellian)

$$p(v) = \frac{1}{Z_q(T_0)} \exp_q\left(-\frac{mv^2}{2k_B T_0}\right). \qquad (6)$$



Important! Constraining the usual expectation $\int dv\, p(v)\left(\frac{mv^2}{2}\right)$ does not work!

Inconsistencies in the use of Tsallis entropy

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Phys. Rev. E **67**, 036114 – **Published 24 March, 2003**

DOI: https://doi.org/10.1103/PhysRevE.67.036114

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Steve Pressé[1,*], Kingshuk Ghosh[2], Julian Lee[3], and Ken A. Dill[4]

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Phys. Rev. Lett. **111**, 180604 – **Published 1 November, 2013**

DOI: https://doi.org/10.1103/PhysRevLett.111.180604

Inconsistencies in the use of Tsallis entropy

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DOI: https://doi.org/10.1103/PhysRevE.90.052149

PAPER

On the foundations of the maximum entropy principle using Fenchel duality for Shannon and Tsallis entropies

Pierre Maréchal, Yasmín Navarrete and Sergio Davis

Published 21 June 2024 • © 2024 IOP Publishing Ltd

Physica Scripta, Volume 99, Number 7

Citation Pierre Maréchal *et al* 2024 *Phys. Scr.* **99** 075265

DOI 10.1088/1402-4896/ad55b8

Physica A: Statistical Mechanics and its Applications
Volume 322, 1 May 2003, Pages 267-275

Superstatistics

C. Beck [a], E.G.D. Cohen [b]

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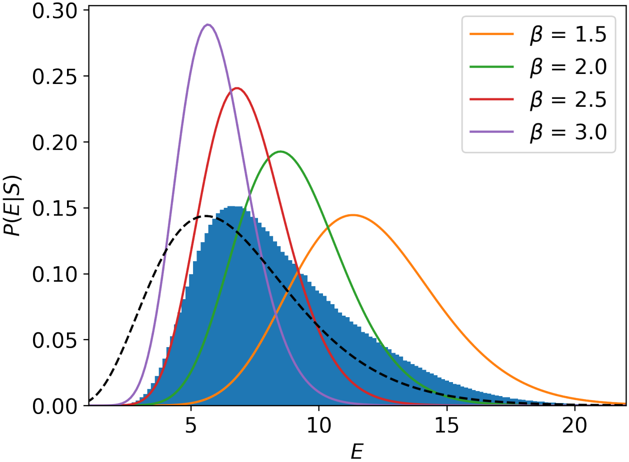
https://doi.org/10.1016/S0378-4371(03)00019-0

Superstatistics replaces the Boltzmann factor $\exp(-\beta E)$ from equilibrium by the mixture

$$\int_0^\infty d\beta\, f(\beta)\exp(-\beta E)$$

of Boltzmann factors at different temperatures weighted by a function $f(\beta) \geq 0$.

An example: a mixture of 4 canonical distributions is not itself a canonical distribution



Moreover, it has the desired *long tails*!

The connection between superstatistics and plasmas

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DOI: https://doi.org/10.1103/PhysRevE.91.012133

Single-particle velocity distributions of collisionless, steady-state plasmas must follow superstatistics

Sergio Davis[1,2,*], Gonzalo Avaria[1,2], Biswajit Bora[1,2], Jalaj Jain[1], José Moreno[1,2], Cristian Pavez[1,2], and Leopoldo Soto[1,2]

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Kamel Ourabah *

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The connection between superstatistics and plasmas

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Stochastic dynamics and superstatistics of the many-particle kappa distribution

E Gravanis[4], E Akylas and G Livadiotis

Published 4 May 2021 •

Journal of Statistical Mechanics: Theory and Experiment, Volume 2021, May 2021

Citation E Gravanis *et al* *J. Stat. Mech.* (2021) 053201

DOI 10.1088/1742-5468/abf7b5

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Physica A: Statistical Mechanics and its Applications

Volume 578, 15 September 2021, 126132

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E. Gravanis, E. Akylas

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The connection between superstatistics and plasmas

Kappa distribution from particle correlations in nonequilibrium, steady-state plasmas

Sergio Davis[1,2,*], Gonzalo Avaria[3], Biswajit Bora[1,2], Jalaj Jain[1], José Moreno[1,2], Cristian Pavez[1,2], and Leopoldo Soto[1,2]

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Phys. Rev. E **108**, 065207 – **Published 8 December, 2023**

DOI: https://doi.org/10.1103/PhysRevE.108.065207

Superstatistics from a dynamical perspective: Entropy and relaxation

Kamel Ourabah[*]

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Phys. Rev. E **109**, 014127 – **Published 22 January, 2024**

DOI: https://doi.org/10.1103/PhysRevE.109.014127

Superstatistics as a full statistical framework

Superstatistics is the unique theory where the inverse temperature $\beta := 1/(k_B T)$ is a random variable with its own probability density $P(\beta|S)$.

- The joint distribution of β with the microstates $\mathbf{\Gamma} = (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_N)$ is

$$P(\mathbf{\Gamma}, \beta|S) = P(\mathbf{\Gamma}|\beta)P(\beta|S) = \left[\frac{\exp\big(-\beta\mathcal{H}(\mathbf{\Gamma})\big)}{Z(\beta)}\right] P(\beta|S). \tag{7}$$

- By integrating out β we arrive at the family of *superstatistical ensembles*

$$P(\mathbf{\Gamma}|S) = \int_0^\infty d\beta P(\beta|S) \left[\frac{\exp(-\beta\mathcal{H}(\mathbf{\Gamma}))}{Z(\beta)}\right] = \rho(\mathcal{H}(\mathbf{\Gamma}); S) \tag{8}$$

  which can be understood as a “deformation” of the canonical ensemble.

- The *ensemble function* $\rho(E; S)$ is given by

$$\rho(E; S) = \int_0^\infty d\beta\, f(\beta; S) \exp(-\beta E) \;\text{ where }\; f(\beta; S) := \frac{P(\beta|S)}{Z(\beta)}.$$

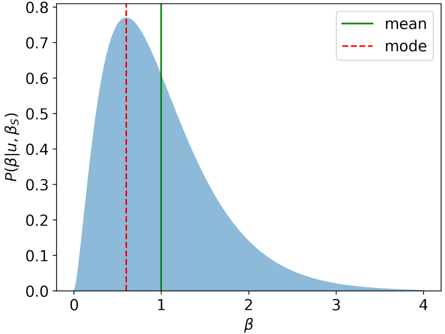
  In other words, it is the Laplace transform of $f(\beta)$.

Superstatistics and the kappa distribution

The kappa distribution follows from superstatistics with $P(\beta|S)$ a gamma distribution,

$$P(\beta|u,\beta_S) = \frac{1}{u\beta_S\,\Gamma(1/u)}\exp\left(-\frac{\beta}{u\beta_S}\right)\left(\frac{\beta}{u\beta_S}\right)^{\frac{1}{u}-1} \qquad 0 \leq u < 1$$

with $\beta_S = \big\langle \beta \big\rangle_{u,\beta_S}$ the mean and $u := \frac{\big\langle (\delta\beta)^2 \big\rangle_{u,\beta_S}}{(\beta_S)^2}$ the relative variance.



The most probable value (mode) is given by

$$\beta^* := \beta_S\,(1-u) \leq \beta_S$$

The original parameters (κ, v_{th}) of the kappa distribution are given in terms of (u, β_S) as

$$\kappa = \frac{1}{u} + \frac{1}{2}$$

$$\frac{m v_{\text{th}}^2}{2} = \frac{1}{(1-u)\beta_S}$$

and we see that $u \to 0$ is equivalent to $\kappa \to \infty$ (Maxwellian). In fact,

$$\lim_{u \to 0} P(\beta | u, \beta_S) = \delta(\beta - \beta_S) \tag{9}$$

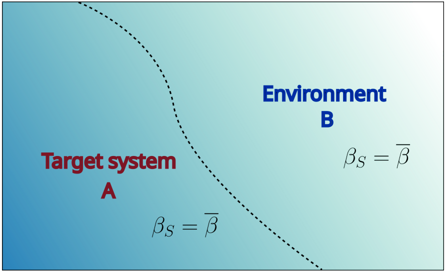
therefore we recover the canonical ensemble as

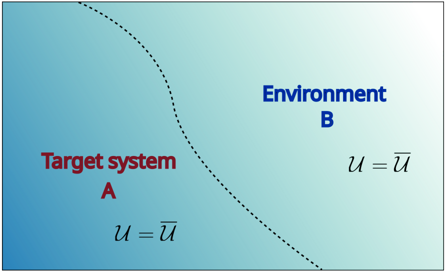
$$\lim_{u \to 0} \rho(E; u, \beta_S) = \lim_{u \to 0} \left[\int_0^\infty d\beta\, f(\beta; u, \beta_S) \exp(-\beta E)\right] = \frac{\exp(-\beta_S E)}{Z(\beta_S)}. \tag{10}$$

Two invariant quantities in non-equilibrium steady states

$$\beta_S := \langle \beta \rangle_S$$

$$\mathcal{U} := \langle (\delta\beta)^2 \rangle_S$$

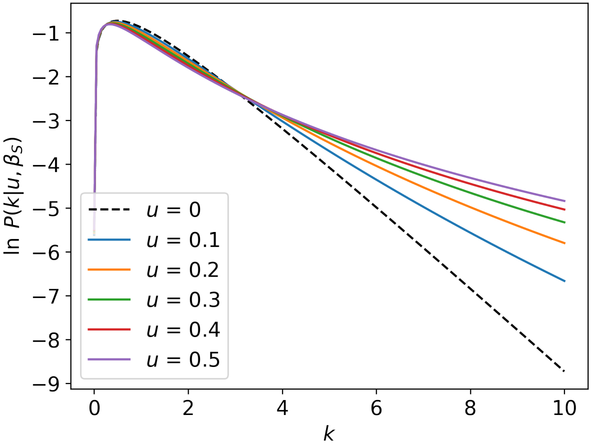




From $\mathcal{U}$ and β_S we can define the dimensionless index

$$u := \frac{\mathcal{U}}{(\beta_S)^2} \tag{11}$$

Distribution of single-particle kinetic energies



$$k = \frac{mv^2}{2} \qquad (12)$$

$$k^* = \frac{1}{2\beta_S(1+u)} \qquad (13)$$

$$P(k|u, \beta_S) = \frac{u\beta_S}{B(3/2, 1/u)} \left[1 + u\beta_S k\right]^{-\left(\frac{1}{u}+\frac{3}{2}\right)} \sqrt{u\beta_S k}$$

Equipartition and uncertainty in kinetic energies

The mean and relative variance of k in the kappa distribution in terms of (u, β_S) are

$$\big\langle k \big\rangle_{u,\beta_S} = \frac{3}{2}\frac{1}{\beta_S\,(1-u)} \tag{14}$$

and

$$\frac{\big\langle (\delta k)^2 \big\rangle_{u,\beta_S}}{\big\langle k \big\rangle^2_{u,\beta_S}} = \frac{2+u}{3(1-2u)} \tag{15}$$

respectively. It follows that $u < 1/2$, and then $\kappa > 5/2$.

- Equipartition holds, not for the mean β_S but for the most probable value β^*
- From the mean and variance of k we can infer u and β_S, thus fitting the distribution
- Higher moments of k may be used to construct statistical tests for detecting kappa distributions

$$\big\langle k^n \big\rangle_{u,\beta_S} = (2\beta_S)^{-n} \prod_{m=1}^{n} \left(\frac{1+2m}{1-mu}\right) \tag{16}$$

PAPER

A superstatistical measure of distance from canonical equilibrium

Sergio Davis

Published 5 July 2024 • © 2024 IOP Publishing Ltd

Journal of Physics A: Mathematical and Theoretical, Volume 57, Number 29

Citation Sergio Davis 2024 *J. Phys. A: Math. Theor.* **57** 295004

DOI 10.1088/1751-8121/ad5caa

The distance from equilibrium in superstatistics turns out to be identical to the *mutual information* between $\boldsymbol{v}$ and β,

$$\mathcal{D}(S) := \left\langle \ln \left[\frac{P(\boldsymbol{v},\beta|S)}{P(\boldsymbol{v}|S)P(\beta|S)} \right] \right\rangle_S. \tag{17}$$

In the case of particles with kappa-distributed velocities,

$$\mathcal{D}(\kappa) = \Phi(\kappa+1) - \Phi(\kappa-1/2) - \frac{3}{2}$$

where $\Phi(z) := \frac{z\Gamma'(z)}{\Gamma(z)} - \ln\Gamma(z)$.

$\mathcal{D}(\kappa)$ decreases monotonically with increasing κ, with $\mathcal{D}=0$ only for $\kappa\to\infty$.

Concluding remarks

- Superstatistics provides an elegant explanation for the origin of kappa distributions, without abandoning the Gibbs-Boltzmann entropy
- The inverse temperature distribution can be written in terms of new, invariant parameters u and β_S, where u measures temperature uncertainty
- The distance from equilibrium in a kappa-distributed plasma has a one-to-one correspondence with the parameter κ (also with u)
- Perhaps kappa distributions are “universal” models that could also be used to describe laboratory (fusion?) plasmas

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